PROBLEM

In the summer of 2011, twenty-four mathematics educators from across Nebraska gathered at the University of Nebraska–Lincoln. These teachers came from both urban and rural districts, from schools that are among the largest in the state to those with only a few mathematics teachers in the entire district. However, all had a common purpose and goal: to raise student achievement in Nebraska, particularly for high-need students, by improving the quality of instruction. Toward this goal, the educators formed a network to cultivate professional growth, support one another in new endeavors, collaborate in creating resources, and share ideas.

A common challenge faced by these educators is heterogeneous classes consisting of learners who struggle alongside those who become bored if not offered thought-provoking tasks. The task confronting teachers is reaching the needs of *all* these learners simultaneously. In addition, teachers are often forced to fit old textbooks to new curricular

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ANALYSIS: CHALLENGING ALL LEARNERS

Teachers can modify existing materials to meet new curricular goals, reaching learners at all levels.

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standards, including the Common Core State Standards for Mathematics (CCSSM). Too often, available materials are outdated, uninteresting, or simply inadequate. Older textbooks rarely provide exercises that thoroughly address the CSSM Standards for Mathematical Practice. One way to address these multiple challenges is problem analysis, a concept we encountered while studying *Mathematics for High School Teachers: An Advanced Perspective* (Usiskin et al. 2003).

WHAT IS PROBLEM ANALYSIS?

Problem analysis is the process of examining a given mathematics exercise to find ways in which the problem can be modified and extended to create a richer learning opportunity for students. Students are often reluctant to attempt what they perceive to be higher-order thinking problems. Problem analysis helps bridge this gap. A teacher can analyze a problem, introduce scaffolding to assist students at the beginning, include intermediate steps to provide

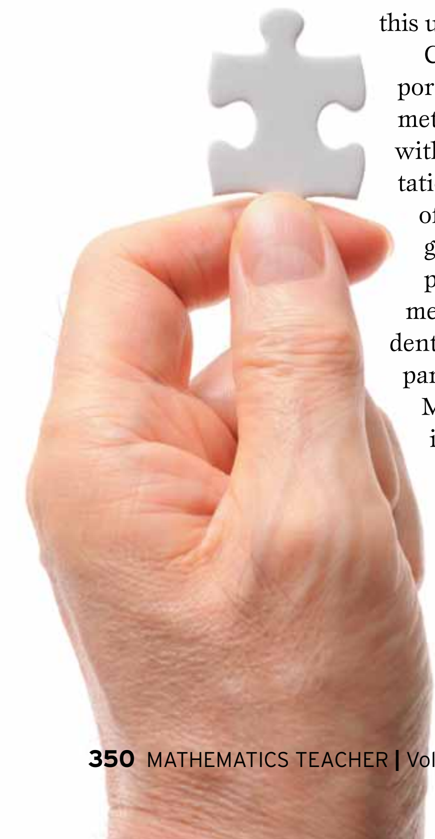
support throughout the problem, and provide extensions to challenge students to go further. In this way, students at all levels can feel challenged and experience success.

GENERALIZING BEYOND EXAMPLES

By contemplating problems presented in their own classroom materials, teachers can use problem analysis to expand the learning opportunities offered by their district-adopted textbook and its ancillary materials. To do so, they should identify an exercise that has potential for deep analysis but merely skims the surface of a mathematically rich task (see **fig. 1**).

Many textbooks emphasize mathematical skills and procedures but present conceptually weak definitions and problems. Such textbooks fall far short of addressing the CCSSM Standards for Mathematical Practices. Using these textbooks as the sole curricular source produces students who are unable to explain and justify mathematical concepts or apply mathematics knowledge to unfamiliar problems or situations. By delving into the richer exercises developed by applying problem analysis to content taken from the district's materials, students use many areas of expertise described in the Standards for Mathematical Practices. These include but are not limited to making sense of problems, reasoning abstractly and quantitatively, constructing viable arguments, and using mathematical modeling.

Certainly, this type of exercise may be new and even uncomfortable for students. It might be new and uncomfortable for their teachers as well. A culture of collaboration and support, both from teachers and classmates, is necessary for students to embrace this unfamiliar mathematical endeavor.



Collaborative pairs or groups can help support learning as students approach this new method of working with mathematics. As with any teaching endeavor, outlining expectations for the appropriate type and amount of help that group members are expected to give is important. Another essential support is familiarizing students with assessment procedures for such exercises. Students may assume that they must master all parts of a task to receive a satisfactory grade. Making students aware that each exercise includes its own benchmarks can ease student anxiety with regard to grading.

When we have introduced these types of problems in the classroom, students have sometimes resisted them, judging them, on first glance, to be "too long." When teachers walk through the first few steps of the problem with students and allow them to experience the various aspects of the problem, students begin to see that there is something in the problem that allows all of them to be successful. After guiding her students through three tasks that were created through problem analysis, co-author Davis reflected, "Although it took some persistence, all but two of my students were actively engaged in making sense of the problems and in acting like mathematicians—exploring the situation, trying out different strategies, and asking each other, 'What if . . . ?' I know the students are building deeper conceptual understanding when I can get them to grapple with the mathematics."

Questions for Problem Analysis

- What happens if I change the parameters of the exercise?
- What happens if I change the question being answered?
- What other mathematical concepts are at work in the exercise that can be highlighted or incorporated?
- How can students generalize their findings to lead them to broader conclusions?
- In what ways can students be asked to modify the problem and explore the concepts in the exercise?
- What insights would I like students to gain regarding this concept?

Fig. 1 If teachers consider careful and thoughtful questions, they can unearth the underlying mathematics.

TILING TUBS

Problem analysis can be implemented for students at all levels of mathematics. Functions are often introduced in introductory algebra courses. Traditional function problems focus on creating tables of values or applying the vertical line test, but these activities do not help students develop the definition of a function. A deep conceptual understanding of functions is necessary for students to make sense of problems and model with mathematics as outlined in the CCSSM Standards for Mathematical Practice.

Finding this curricular weak point creates an opportunity for problem analysis. Using the Tiling Tubs problem (Papick 2007) and its analysis rather than using traditional input-output table exercises allows students to discover the definition of a function by modeling a real-life situation (see **fig. 2** for the problem and see the **appendix** for solutions).

Initially, the focus of the problem is simply finding a specific number of tiles. When students are asked to generalize their solution, they are encouraged to stretch their thinking and add a level of abstraction. Having students carefully consider which values for s and n make sense for the

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INPUT-OUTPUT TABLES Make an input-output table for the function. Use 0, 1, 2, and 3 as the domain.

13. $y = 3x + 2$ 14. $y = 21 - 2x$ 15. $y = 5x$

16. $y = 6x + 1$ 17. $y = 2x + 1$ 18. $y = x + 4$

Source: Larson et al. (2001, p. 49)

The Tiling Tubs Problem

A square hot tub with sides of length 4 feet is surrounded by square border tiles. The border tiles measure 1 foot on each side. How many tiles are going to be needed for the border?

Problem Analysis

(*a*) Answer the question, letting n be the number of tiles.

(*b*) Generalize the question by replacing 4 feet with s feet. (It might help to make sketches on grid paper to find the number of tiles needed for hot tubs of various side lengths. Create a table of your data and use this information to develop a rule for finding the number of tiles needed for any side length.) Write an equation that answers the question for all s.

(*c*) Graph your rule so that s is on the x-axis and n is on the y-axis. Interpret the graph in terms of the original question. What values of s make sense? What values of n make sense?

(*d*) Is the relationship linear? Explain how your answer is supported by both the table and the graph.

(*e*) Now, solve your equation for s. Graph your rule so that n is on the x-axis and s is on the y-axis. Interpret the graph in terms of the original question. What values of n make sense? What values of s make sense?

(*f*) Write at least one more equation for the number of tiles needed for the border of the hot tub. How could you convince someone that your expressions for the number of tiles needed are equivalent?

(*g*) Is the number of tiles needed a function of the side length of the hot tub?
- What is the domain of the function?
- What is the co-domain of the function?
- What is the range of the function?
- Is the function one-to-one?

Fig. 2 The Tiling Tubs problem can serve as an introduction to functions for prealgebra students

problem provides the foundation for analyzing the domain and range of more advanced functions in future courses. Next, students are asked to analyze the type of relationship demonstrated in the problem by looking at both a table of values and a graph. Here, creating the table of values is explicitly tied to the meaning of the dependent and independent variables, which addresses reasoning and sense making in a way that traditional problems do not.

In a class with learners at all levels, scaffolding at the beginning of an exercise is vital to provide some support so that all students can experience success. As a problem becomes more sophisticated, the level of thinking and the complexity of the mathematics also advance. The modified problem is an example of low-threshold, high-ceiling problems that allow entry points for struggling students while maintaining rich mathematical ideas to challenge all students.

Teachers can assess such a problem by creating benchmarks within the problem. For example, a student who successfully completes the original problem may be scored at a C (or basic) level, whereas a student who navigates through the entire exercise and can analyze the broader connections may be scored at the A (or advanced) level. We have found that informing students of these benchmarks allows learners to feel successful even if they are unable to complete all parts of a problem. Students are encouraged to keep working toward the problem's more challenging aspects while feeling some security about their course grade. Co-author Garcia notes, "Initially, several students were concerned with how these nonstandard tasks would impact their grade in class. It took some time, but those students came to see that they did not have to be successful in every part of each problem to achieve success and a 'good' grade."

INVESTIGATING NEGATIVE AND ZERO EXPONENTS

As illustrated above, problem analysis allows teachers to take an example or exercise from the textbook and create a deeper learning experience for students. The rules of exponents, often presented for students to memorize, offer teachers an opportunity for problem analysis. Students can experience and develop the rules for themselves by completing a well-constructed task. This speaks to the CCSSM practice of looking for and expressing regularity in repeated reasoning. By having students investigate the concept of negative and zero exponents through a real-world example rather than simply filling in a table, learners are able to develop a conceptual understanding of the role of negative and zero exponents and their application outside the mathematics classroom (see **fig. 3** for the problem; go to **www.nctm.org/mt048** for solutions).

A VALUABLE TEACHING TOOL

Procedural skill and conceptual understanding are both important in developing strong mathematics students. By implementing problem analysis, teachers

ACTIVITY

Developing Concepts

Investigating Zero and Negative Exponents

1. Copy the table and discuss any patterns you see. Use the patterns to complete the table. Write non-integers as fractions in simplest form.

Exponent, n	3	2	1	0	−1	−2	−3
Power, 2^n	8	4	2	?	?	?	?
Power, 3^n	27	9	3	?	?	?	?
Power, 4^n	64	16	4	?	?	?	?

2. What appears to be the value of a^0 for any number a?
3. How can you evaluate an expression of the form a^{-n}?

DEFINITION OF ZERO AND NEGATIVE EXPONENTS

Let a be a nonzero number and let n be a positive integer.
- A nonzero number to the zero power is 1: $a^0 = 1$, $a \neq 0$.
- a^{-n} is the reciprocal of a^n: $a^{-n} = \frac{1}{a^n}$, $a \neq 0$.

Source: Larson et al. (2001, p. 456)

Investigating Negative and Zero Exponents

Your friend Fred has been selected to be a contestant on the hot new game show *Power of Math.* He starts the game with \$1000. Each time he answers a question correctly, he doubles his money. Each time he answers a question incorrectly, his money is cut in half.

Problem Analysis

(*a*) What happens if Fred answers the same number of questions correctly that he answers incorrectly?

(*b*) Find at least five different ways in which Fred could win 4 times his starting amount (4 • \$1000 = \$4000).

(*c*) Find at least five different ways in which Fred could win 16 times his starting amount (16 • \$1000 = \$16,000).

(*d*) Find at least five different ways in which Fred could win half his starting amount (1/2 • \$1000 = \$500).

(*e*) What happens if Fred answers more questions incorrectly than correctly? Could he ever end up with a negative amount in this game? Explain.

(*f*) Complete the table below to analyze Fred's winnings.

	3 More Correct Than Incorrect	2 More Correct Than Incorrect	1 More Correct Than Incorrect	Same Correct as Incorrect	1 More Incorrect Than Correct	2 More Incorrect Than Correct
Amount Won						
Number of Times Fred's Starting Amount						
Amount Won as a Power of 2 Times Fred's Starting Amount						

(*g*) If Fred answers *m* questions correctly and *n* questions incorrectly, what is his ending amount? Describe this ending amount using a mathematical expression or equation.

Fig. 3 This example helps students to develop a conceptual understanding of positive and negative exponents.

are able to take skills-based problems from existing textbooks and create rich mathematical tasks. Many professional mathematicians do not fit the mold of what traditional classrooms deem good mathematics students because they always seem to be asking "What if?" and thinking beyond what is asked in exercises. By exposing students to tasks using well-designed problem analysis, students are able to uncover and hone their abilities to reason and solve problems. With appropriate procedures in place and with an established culture of perseverance when problem solving, problem analysis can be a valuable teaching tool to reach learners of all levels.

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